Recovering the Spectrum of a Narrow-band Process From Syncopated Samples

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Losslessly sampling a band-limited narrow-band process at an average rate equal to the Nyquist rate may require a nonuniform sampling strategy. One such strategy is phase quadrature sampling, in which a process of bandwidth B is sampled at rate B in each of two channels where the two channels are π/2 out of phase at frequency B. Phase quadrature sampling is a special case of syncopated sampling, where the phase between channels is fixed but arbitrary. A simple method for recovering the spectrum of the input process from syncopated samples is derived. The derivation indicates what values of phase between channels result in lossless sampling.

I. Introduction

Band-limited narrow-band processes can be losslessly sampled at rates which correspond to the Nyquist rate for the passband width rather than a rate of twice the highest frequency present. This is accomplished by sampling at periodic but nonuniform intervals (Refs. 1, 2 and 3). Time domain interpolation formulas are known for recovering the original process from such samples (Refs. 1 and 2). The general approach is based upon syncopated sampling, where a process of bandwidth B is sampled in two channels at a uniform rate B in each channel and a phase offset θ between channels. With phase quadrature sampling, θ is equal to $\pi/2$, or one-fourth, of a sample interval. A simple method for recovering the spectrum from two channels of samples is derived.

The current motivation for studying syncopated sampling is the Wide Band Data Acquisition System (WBDAS), which is used to collect data for Very Long Baseline Interferometry (VLBI) clock synchronization measurements (Ref. 3). The signal entering WBDAS from a station receiver is centered at 50 MHz and has been bandpass filtered so that most of its energy is in the range from 25 to 75 MHz. The WBDAS samples, quantizes, and records this signal.

In the WBDAS, phase quadrature sampling is employed. What this does to the signal is similar to what is done by single sideband demodulation, but without the side effect of signal group delay from an analog demodulator. The narrow-band signal $s(t)$ going into the sampler can be expressed in terms of cosine and sine components, $x(t)$ and $y(t)$, relative to the $\omega_o = 2\pi$ (50 MHz) center frequency:

$$s(t) = x(t)\cos(\omega_o t) + y(t)\sin(\omega_o t) \tag{1}$$

Both $x(t)$ and $y(t)$ are baseband processes of bandwidth 25 MHz. The in-phase channel samples are taken at times nT

and the quadrature phase channel samples are taken at times $(n + 1/4)T$ where n is an integer and T is 20 nsec. The in-phase channel samples are:

$$
\begin{aligned}
s(nT) &= x(nT) \cdot 1 + y(nT) \cdot 0 \\
&= x(nT)
\end{aligned}
\tag{2}
$$

and the quadrature channel samples are:

$$
\begin{aligned}
s\left[\left(n+\frac{1}{4}\right)T\right] &= x\left[\left(n+\frac{1}{4}\right)T\right] \cdot 0 + y\left[\left(n+\frac{1}{4}\right)T\right] \cdot 1 \\
&= y\left[\left(n+\frac{1}{4}\right)T\right]
\end{aligned}
\tag{3}
$$

The two channels contain uniform samples at the Nyquist rate for $x(t)$ and $y(t)$, respectively. The quadrature sampling scheme can thus be intuitively viewed as a combination of demodulation and uniform sampling.

Uniformly sampling $s(t)$ at a rate equal to the average rate of the phase quadrature scheme will result in the loss of information. Sampling at times nT, where T is 10 nsec, gives the following result from Eq. (1):

$$s(nT) = x(nT)(-1)^n \tag{4}$$

The result is that while $x(t)$ has been sampled at a rate of 100 MHz or twice the Nyquist rate for a 25 MHz baseband signal, $y(t)$ has been completely lost.

II. Recovering the Spectrum

The spectrum of a band-limited narrow-band process of bandwidth B will be recovered from two channels of uniform samples taken at rate B where the two channels are offset by a phase θ. Being able to recover the spectrum indicates that the sampling is lossless as the process can be recovered by taking the inverse Fourier transform of its spectrum. This syncopated sampling scheme gives us the spectrum for any center frequency of the narrow-band process. One restriction is that given the center frequency, there are values of θ for which the method breaks down; these values are specified here.

The input signal $x(t)$ is sampled by two impulse trains $s_1(t)$ and $s_2(t)$ where:

$$s_1(t) = \sum_{m=-\infty}^{\infty} T\delta(t - mT) \tag{5}$$

$$s_2(t) = \sum_{m=-\infty}^{\infty} T\delta[t - (mT + T_o)] \tag{6}$$

T = sampling interval for both channels

T_o = time offset between the two channels

The resulting sampled signals on the two sampling channels are:

$$x_1(t) = s_1(t)x(t) \tag{7}$$

and

$$x_2(t) = s_2(t)x(t) \tag{8}$$

Next, the effect of sampling in the frequency domain is considered. Fourier transform pairs are denoted as follows, using capitals in the frequency domain:

$$X(f) \leftrightarrow x(t)$$

$$X_1(f) \leftrightarrow x_1(t)$$

$$X_2(f) \leftrightarrow x_2(t)$$

The sampling functions can be written in the time domain as a sum of exponentials:

$$s_1(t) = \sum_{m=-\infty}^{\infty} e^{2\pi j m t/T} \tag{9}$$

$$s_2(t) = \sum_{m=-\infty}^{\infty} e^{-jm\theta} e^{2\pi j m t/T} \tag{10}$$

where $\theta = 2\pi T_o/T$

The frequency translation theorem $X(f + f_o) \leftrightarrow x(t)e^{-2\pi j f_o t}$ results in:

$$X_1(f) = \sum_{m=-\infty}^{\infty} X\left(f + \frac{m}{T}\right) \tag{11}$$

$$X_2(f) = \sum_{m=-\infty}^{\infty} e^{-jm\theta} X\left(f + \frac{m}{T}\right) \tag{12}$$

The purpose of this discussion is to describe circumstances under which $X(f)$ can be recovered from $X_1(f)$ and $X_2(f)$. Suppose $x(t)$ is band-limited to the range $f_c < f < f_c + 1/T$ and

$-1/T - f_\ell < f < -f_\ell$. Then for $f_\ell < f < f_\ell + 1/T$, Eqs. (11) and (12) reduce to:

$$X_1(f) = X(f) + X\left(f + \frac{\hat{m}}{T}\right) \tag{13}$$

$$X_2(f) = X(f) + e^{-j\hat{m}\theta}\, X\left(f + \frac{\hat{m}}{T}\right) \tag{14}$$

where $\hat{m}$ in an integer function of f satisfying:

$$-f_\ell - \frac{1}{T} < f + \frac{\hat{m}}{T} < -f_\ell \tag{15}$$

It then follows that:

$$\begin{aligned} &\text{(i) } X(f) = \frac{X_1(f) - e^{j\hat{m}\theta}\, X_2(f)}{1 - e^{j\hat{m}\theta}} \text{ for } f_\ell < f < f_\ell + \frac{1}{T} \\ &\text{(ii) } X(f) = X^*(-f) \text{ for } -f_\ell - \frac{1}{T} < f < -f_\ell \\ &\text{(iii) } X(f) = 0 \text{ elsewhere} \end{aligned} \tag{16}$$

where * denotes complex conjugation. Property (ii) is a consequence of $x(t)$ being real valued.

III. Examples

In this section, the method for computing $X(f)$ is applied to specific cases. First $X_1(f)$ and $X_2(f)$ are expressed in terms of the samples:

$$\begin{aligned} X_1(f) &= \int_{-\infty}^{\infty} x_1(t)\, e^{-2\pi jft}\, dt \\ &= T \sum_{m=-\infty}^{\infty} x(mT)\, e^{-2\pi jfmt} \end{aligned} \tag{17}$$

$$\begin{aligned} X_2(f) &= \int_{-\infty}^{\infty} x_2(t)\, e^{-2\pi jft}\, dt \\ &= T e^{-j\theta fT} \sum_{m=-\infty}^{\infty} x(mT + T_o)\, e^{-2\pi jfmT} \end{aligned} \tag{18}$$

A. Uniform Sampling

The first case is uniform sampling on one channel of a baseband signal of bandwidth $1/T$. The sampling rate is $2/T$, the Nyquist rate. This system can be expressed as two channels of sampling run at rate $1/T$, offset in time by $T/2$. Equation (16) reduces to

$$X(f) = \frac{1}{2}\,[X_1(f) + X_2(f)] \tag{19}$$

which by Eq. (17) and (18) results in

$$X(f) = \frac{T}{2} \sum_{n=-\infty}^{\infty} x\left(n\frac{T}{2}\right) e^{-2\pi jfn\frac{T}{2}} \tag{20}$$

which, in fact, corresponds to uniform sampling at rate 2/T on one channel.

B. Quadrature Phase Sampling

The second special case is quadrature phase sampling. Here, the sampling period on each of two channels is T, and the sampling times on the two channels are offset by $T/4$. This type of sampling is applicable to narrow-band signals of bandwidth $1/T$ centered about frequency $1/T$ (or some integer multiple of $1/T$). Considering the signal to be centered about $1/T$:

$$f_\ell = \frac{T}{2} \qquad \frac{T}{2} < f < \frac{3T}{2} \qquad T_o = \frac{T}{4}$$

$$\hat{m} = -2 \qquad \theta = 2\pi\frac{T_o}{T} = \frac{\pi}{2}$$

which by Eq. (16) results in

$$X(f) = \frac{1}{2}[X_1(f) + X_2(f)] \tag{21}$$

Employing Eqs. (17) and (18) to express $X_1(f)$ and $X_2(f)$ in terms of the samples gives:

$$\begin{aligned} X(f) = \frac{T}{2}\Bigg\{ &\sum_{-\infty}^{\infty} x(mT)\, e^{-2\pi jfmT} \\ &+ \sum_{-\infty}^{\infty} x\left[\left(m + \frac{1}{4}\right)T\right] e^{-2\pi jf\left(m + \frac{1}{4}\right)T} \Bigg\} \\ \frac{T}{2} < f < \frac{3T}{2} \end{aligned} \tag{22}$$

For the channel offset T_o differing from $T/4$, one can still formulate $X(f)$ in terms of $X_1(f)$ and $X_2(f)$ by way of Eq.

(16). The quantity θ will differ from $\pi/2$ and the formula will be more complicated than Eq. (21). When the sample offset is $T_0 = T/2$, then $\theta = \pi$ and Eq. (16) becomes singular as $1 - e^{j\hat{m}\theta}$ becomes zero. Since this corresponds to uniform sampling on one channel, uniform sampling will not work for this passband at the average sample rate used for quadrature phase or syncopated sampling schemes near quadrature phase.

Finally, taking the inverse Fourier transform of Eq. (22) gives the sample reconstruction formula for quadrature phase sampling:

$$x(t) = \sum_{-\infty}^{\infty} x(mT) \frac{\sin\left(\frac{\pi}{T}(t - mT)\right)}{\frac{\pi}{T}(t - mT)} \cos\frac{2\pi t}{T}$$

$$+ \sum_{-\infty}^{\infty} x\left(mT + \frac{T}{4}\right) \frac{\sin\left\{\frac{\pi}{T}\left[t - \left(m + \frac{1}{4}\right)T\right]\right\}}{\frac{\pi}{T}\left[t - \left(m + \frac{1}{4}\right)T\right]} \sin\frac{2\pi t}{T} \tag{23}$$

This formula has been obtained before by working directly in the time domain (see Ref. 2).

References

1. Woodward, P. M., *Probability and Information Theory with Applications to Radar*, pp. 34-35, McGraw-Hill, New York, 1955.
2. Grace, O. D., and Pitt, S. P., "Quadrature Sampling of High-Frequency Waveforms," *Journal of the Acoustical Society of America*, Vol. 44, No. 5, pp. 1453-1454, 1968.
3. Hurd, W. J., "Preliminary Demonstration of Precision DSN Clock Synchronization by Radio Interferometry," *Deep Space Network Progress Report 42-37* pp. 57-68, Jet Propulsion Laboratory, Pasadena, California, February 15, 1977.
4. Ahmed, N., and Rao, K. R., *Orthogonal Transforms for Digital Signal Processing*, Springer-Verlag, Berlin, Heidelberg, New York, 1975.
5. Oppenheim, A. V., and Schafer, R. W., *Digital Signal Processing*, Prentice-Hall, Englewood Cliffs, New Jersey, 1975.